Classical transverse Ising spin glass with short-range interactions beyond the mean-field approximation

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Abstract. The classical transverse field Ising spin-glass model with short-range interactions is investigated beyond the mean-field approximation for a real d-dimensional lattice. We use an appropriate non-trivial modification of the Bethe–Peierls method recently formulated for the Ising spin-glass. The zero-temperature critical value of the transverse field and the linear susceptibility in the paramagnetic phase are obtained analytically as functions of dimensionality d. The phase diagram is also calculated numerically for different values of d. In the limit $d \to \infty$, known mean-field results are consistently reproduced.

The study of glasses is today one of the most relevant and actual problems in condensed matter physics. Originally, the basic idea was to start from spin-glass (SG) models and to extract as much as was possible at a mean-field-approximation (MFA) level [1–4]. However, there have been recent studies [5–8] which have indicated difficulties in extending the MFA scenario to realistic spin glasses with short-range interactions and deciding *a priori* which properties survive and which must be appropriately modified. Renormalization group treatments [9–11] for classical and quantum spin glasses and phenomenological studies [12] do not seem to suggest a clear picture.

Quite recently, in an interesting paper [13], an approach beyond the MFA has been achieved for a d-dimensional Ising SG model with short-range interactions on a real lattice using an extension of the Bethe–Peierls approximation (BPA) [14] to the spin-glass problem via the replica trick. This approach seems to be very promising for establishing a direct link with the results obtained by different authors for the infinite-ranged version and for controlling possible deviations for short-ranged glasses from the well acquired MFA scenario. Of course, additional applications to more complex glassy systems and improvements are necessary in order to understand something more about the role played by the glassy fluctuations around the MFA solution for finite dimensionalities.

In this paper we explore the glassy properties of the d-dimensional classical transverse field Ising SG model [15–17] with short-range interactions using an appropriate non-trivial modification of the BPA formulated originally in [13] for the Ising SG. The model considered here has received recent attention because it is a relatively simple SG model which reflects some properties of the quantum counterpart [4] and it is a specific example of a classical two-vector anisotropic SG. Therefore, the classical limit of the usual more complex quantum Ising SG model in a transverse field with realistic exchange interactions may be useful for

making contact [16, 17] with low-temperature properties of the so-called 'proton glasses' [15], such as the compounds $Rb_{1-x}(NH_4)_xH_2PO_4$, and with the most recent experimental magnetic data for the dipolar glass $Li\,Ho_{0.167}Y_{0.822}F_4$ [18, 19].

As concerning the quantum realistic SG model, a lot of results have been obtained only for $d = 1$ [20, 21] and for infinite-ranged interactions ($d = \infty$) [4, 19]. For the classical counterpart, only results with infinite-ranged interactions have been derived [16, 17], in particular at $T = 0$, and even in the 'simple' MFA limit the full phase diagram has not yet been calculated. In any case, there are few studies about short-ranged glassy models. It, therefore, appears quite relevant that the BPA allows us to describe some non-trivial glassy properties of the model (in the paramagnetic phase) for $d \geqslant 1$. In particular, explicit analytical results are obtained at $T = 0$ and the phase diagram in the (temperature, transverse field)-plane is derived numerically for an arbitrary dimensionality.

The classical transverse field SG model considered here is described by the Hamiltonian [15–17]

$$H = -\frac{1}{2} \sum_{\langle i,j \rangle} J_{ij} S_i^z S_j^z - \Gamma \sum_{i=1}^{N} S_i^x - h \sum_{i=1}^{N} S_i^z \tag{1}$$

with $(S_i^z)^2 + (S_i^x)^2 = 1$. Here Γ and h are transverse and longitudinal fields, respectively, and the couplings J_{ij} are independent random variables assuming values $\pm J$ with equal probability. In (1) $\sum_{\langle i,j \rangle} \ldots$ denotes a sum over nearest-neighbour pairs of N sites on a hypercubic d-dimensional lattice. Using the replica trick, the problem is reduced to determining the 'quenched average':

$$Z_n = \left[\mathrm{Tr} \exp \left(-\beta \sum_{\alpha=1}^{n} H_\alpha \right) \right]_{\mathrm{av}} \tag{2}$$

where H_α is the αth replica of the Hamiltonian (1) and $\beta = 1/T$ with the Boltzmann constant $k_B \equiv 1$. Working directly on the real lattice, the basic idea of the BPA for spin glasses [13] is to take into account the correct interactions inside replicated clusters (cl), constituted by a central spin S_0 and its $2d$ nearest neighbours $\{S_i; i = 1, \ldots, 2d\}$, and to describe the interactions of the cluster borders with the remnant (rm) of the system by means of effective couplings among replicas to be determined self-consistently. With this in mind, equation (2) for the Bethe–Peierls ansatz can be formally rewritten as [13]

$$Z_n = \mathrm{Tr}_{\{S_{\mathrm{cl}}\}} \left[\exp \left(-\beta \sum_{\alpha=1}^{n} H_\alpha^{(\mathrm{cl})} \right) \mathrm{Tr}_{\{S_{\mathrm{rm}}\}} \exp \left(-\beta \sum_{\alpha=1}^{n} H_\alpha^{(\mathrm{rm})} \right) \right]_{\mathrm{av}} \tag{3}$$

$$\equiv K(T, \Gamma, h) \mathrm{Tr}_{\{S_{\mathrm{cl}}\}} [\exp(-\beta \mathcal{H}_n)]_{\mathrm{av}}$$

where

$$H_\alpha^{\mathrm{cl}} = -\Gamma \sum_{k=0}^{2d} S_{k\alpha}^x - \sum_{i=1}^{2d} J_{0i} S_{0\alpha}^z S_{i\alpha}^z \tag{4}$$

and $H_\alpha^{(\mathrm{rm})}$ denotes replicated Hamiltonians of the cluster and remnant of the system interacting with cluster borders, respectively, and $K(T, \Gamma, h)$ is a multiplicative constant independent of lateral spins,

$$\mathrm{Tr}_{\{S_{\mathrm{cl}}\}} \ldots = \frac{1}{\pi^{n(2d+1)}} \int_{-1}^{1} \prod_{k=0}^{2d} \prod_{\alpha=1}^{n} \frac{\mathrm{d}S_{k\alpha}^z}{\sqrt{1 - (S_{k\alpha}^z)^2}} \ldots \tag{5}$$

and

$$\mathcal{H}_n = -\sum_{\alpha=1}^{n}\sum_{i=1}^{2d} J_{0i} S_{0\alpha}^z S_{i\alpha}^z - \frac{\beta J^2}{2}\sum_{\alpha,\alpha'=1}^{n}\sum_{i=1}^{2d}\lambda_{\alpha\alpha'} S_{i\alpha}^z S_{i\alpha'}^z$$

$$-\frac{1}{\beta}\sum_{\alpha=1}^{n}\sum_{k=0}^{2d}\ln\cosh[\beta\Gamma\sqrt{1-(S_{k\alpha}^z)^2}] - h\sum_{\alpha=1}^{n}\sum_{i=1}^{2d}S_{k\alpha}^z \tag{6}$$

with $\lambda_{\alpha\alpha'} = \mu_{\alpha\alpha'}$ for $\alpha \neq \alpha'$ and $\lambda_{\alpha\alpha} = \mu$, which are parameters to be determined via appropriate self-consistent equations. Here we have used the relation $S_{k\alpha}^x = \pm(1-(S_{k\alpha}^z)^2)^{1/2}$ ($k = 0, 1, \ldots, 2d$). Of course, if a transition from a paramagnetic phase to a SG one is assumed to exist, one expects $\mu_{\alpha\alpha'} = 0$ in the paramagnetic phase.

At this stage, the self-consistent equations which determine the effective couplings $\mu_{\alpha,\alpha'}$ and μ as $n \to 0$ are

$$\langle S_{i\alpha}^z S_{i\alpha'}^z\rangle = \langle S_{0\alpha}^z S_{0\alpha'}^z\rangle \qquad \text{with } i = 1, \ldots, 2d \tag{7}$$

where

$$\langle\cdots\rangle = \frac{\text{Tr}[\exp(-\beta\mathcal{H}_n)\ldots]_{\text{av}}}{\text{Tr}[\exp(-\beta\mathcal{H}_n)]_{\text{av}}}. \tag{8}$$

It is easy to check that, for $h = 0$, due to the inversion symmetry $S_{i\alpha}^z \to -S_{i\alpha}^z$ and symmetry of the probability distribution for J_{ij}, equation (7) with $\alpha = \alpha'$ can be reduced to the following:

$$\chi_i = \chi_0 \qquad \text{for } i = 1, \ldots, 2d \tag{9}$$

where

$$\chi_k = \left.\frac{\partial\langle S_{k\alpha}\rangle}{\partial h}\right|_{h=0} \qquad \text{with } k = 0, 1, \ldots, 2d \tag{10}$$

denotes the local susceptibility.

We are now in a position to obtain the explicit equations for μ which will also be used to obtain the phase diagram of the model. Since it is expected that $\mu_{\alpha,\alpha'} \to 0$ approaching the spin-glass transition from below, for $n \to 0$, one obtains at $h = 0$

$$\langle S_{k\alpha}^z S_{k\alpha'}^z\rangle = \beta J^2 \mu_{\alpha\alpha'}\sum_{i=1}^{2d}[\langle S_k^z S_i^z\rangle_0]_{\text{av}} + \mathcal{O}(\mu_{\alpha,\alpha'}^2) \qquad (k = 0, 1, \ldots, 2d) \tag{11}$$

where

$$\langle\cdots\rangle_0 = \frac{1}{\pi^{2d+1}Z_0}\int_{-1}^{1}\prod_{k=0}^{2d}\frac{\mathrm{d}S_k^z}{\sqrt{1-(S_k^z)^2}}\exp(-\beta\mathcal{H}_0)\ldots \tag{12}$$

with

$$\mathcal{H}_0 = -\sum_{i=1}^{2d}J_{0i}S_0^z S_i^z - \frac{\beta J^2}{2}\mu\sum_{i=1}^{2d}(S_i^z)^2 - \frac{1}{\beta}\sum_{k=0}^{2d}\ln\cosh[\beta\Gamma\sqrt{1-(S_k^z)^2}]. \tag{13}$$

In equation (12) Z_0 denotes the normalization factor. The term $-h\sum_{k=0}^{2d}S_k^z$ must be added to the right-hand side of (13) when it is necessary. So, at $h = 0$ due to the translational symmetry for the sample-averaged system and assuming $\mu_{\alpha\alpha'} = 0$ at and above the glassy transition line (to be determined), the self-consistent equation (7) for $\alpha \neq \alpha'$ and $\alpha = \alpha'$ reduces, respectively, to

$$[\langle(S_i^z)^2\rangle_0^2]_{\text{av}} + (2d-1)[\langle S_i^z S_j^z\rangle_0^2]_{\text{av}} = 2d[\langle S_0^z S_j^z\rangle_0^2]_{\text{av}} \qquad (i \neq j = 1, \ldots, 2d) \tag{14}$$

and

$$[\langle (S_i^z)^2 \rangle_0]_{av} = [\langle (S_0^z)^2 \rangle_0]_{av} \qquad (i = 1, \dots, 2d) \tag{15}$$

where $i \neq j$ denote arbitrary lateral sites of the cluster with the central spin S_0.

By solving equations (14) and (15), it is possible to obtain the phase diagram of our model in the (T, Γ) plane. Explicit results can be derived analytically only at $T = 0$. As $T \to 0$, introducing $\overline{\mu} = \beta\mu$ which is finite, with the help of equation (12) and choosing $i = 1$ in equation (15), one obtains

$$\overline{\mu} = \frac{1}{J^2} \frac{\Gamma - \sqrt{\Gamma^2 - 4(2d-1)J^2}}{2}. \tag{16}$$

Hence taking into account equation (9) one finds, in the paramagnetic phase at $T = 0$, the linear susceptibility

$$\chi = 2 \frac{\Gamma + \sqrt{\Gamma^2 - 4(2d-1)J^2}}{(\Gamma + \sqrt{\Gamma^2 - 4(2d-1)J^2})^2 - 4J^2}. \tag{17}$$

Now, we calculate the critical value Γ_c at $T = 0$ of the transverse field using equations (14) and (16). For $i = 1$ and $j \neq 1$, with some algebra we rewrite (14), as $T \to 0$, in the following form:

$$(\Gamma_c - \overline{\mu}_c J^2)^4 - 2dJ^2(\Gamma_c - \overline{\mu}_c J^2)^2 + (2d-1)J^4 = 0 \tag{18}$$

with

$$\overline{\mu}_c J^2 = \tfrac{1}{2}[\Gamma_c - \sqrt{\Gamma_c^2 - 4(2d-1)J^2}].$$

From this equation one easily obtains

$$\Gamma_c = 2(2d-1)^{1/2}J. \tag{19}$$

As we see, χ is positive and has physical meaning only for $\Gamma \geqslant \Gamma_c$. This suggests that the expression (17) for χ is related only to the paramagnetic phase.

For $(\Gamma - \Gamma_c)/\Gamma_c \ll 1$ equation (17) yields

$$\chi \approx \begin{cases} \dfrac{(2d-1)^{1/2}}{2J(d-1)}\left[1 - \dfrac{d\sqrt{2}}{d-1}\left(\dfrac{\Gamma - \Gamma_c}{\Gamma_c}\right)^{1/2} + \mathcal{O}\left(\dfrac{\Gamma - \Gamma_c}{\Gamma_c}\right)\right] & \text{for } d \neq 1 \\[4mm] \dfrac{1}{2J\sqrt{2}}\left(\dfrac{\Gamma - \Gamma_c}{\Gamma_c}\right)^{-1/2}\left[1 + \mathcal{O}\left(\dfrac{\Gamma - \Gamma_c}{\Gamma_c}\right)^{1/2}\right] & \text{for } d = 1. \end{cases} \tag{20}$$

As an 'a posteriori' justification of the correctness of the glassy BPA (3), it is easy to check analytically that using the rescaling $J \to J/\sqrt{2d}$ one finds $\overline{\mu} = 2d\chi$ and we get at $T = 0$ for $d \to \infty$

$$\chi = \frac{1}{2J^2}[\Gamma - (\Gamma^2 - \Gamma_c^2)^{1/2}] \qquad \text{with} \qquad \Gamma_c = 2J + \mathcal{O}(d^{-1/2}).$$

These results reproduce exactly those obtained at $T = 0$ for the same SG model but with infinite-ranged interactions [16]. This partial result supports the validity of the BPA for SGs.

The situation for $d = 1$ with a divergence of the linear susceptibility at $T = 0$ as $\Gamma \to \Gamma_c^+$ can be simply explained. With the dichotomic probability distribution of one-dimensional nearest-neighbour couplings $J(i, i+1) \equiv J_i$ ($J_i = \pm J$ with $J > 0$), after the gauge transformation of spin variables

$$S_i^z \to \text{sign}(J_1) \dots \text{sign}(J_{i-1}) S_i^z \tag{21}$$

the system can be reduced to the uniform ferromagnet in an external transverse field Γ. Therefore, it is natural to expect that at $\Gamma = \Gamma_c$ the ferromagnetic phase transition with a divergent linear susceptibility occurs. Indeed, a more detailed analysis of the one-dimensional case shows that at $T = 0$ the linear susceptibility χ can be calculated exactly for the paramagnetic phase. The divergence of χ is the same as that obtained within the BPA for the one-dimensional system.

For arbitrary T and Γ, from equations (14) and (15) one can calculate numerical equilibrium properties of our model in the paramagnetic phase. In particular, in figure 1 we show the phase diagram in the (Γ, T)-plane for different d. We have conveniently scaled variables T and Γ to reproduce the results at a very high dimensionality. In figure 2 a variation of the rescaled critical temperature with a dimension at $\Gamma = 0$ is plotted.
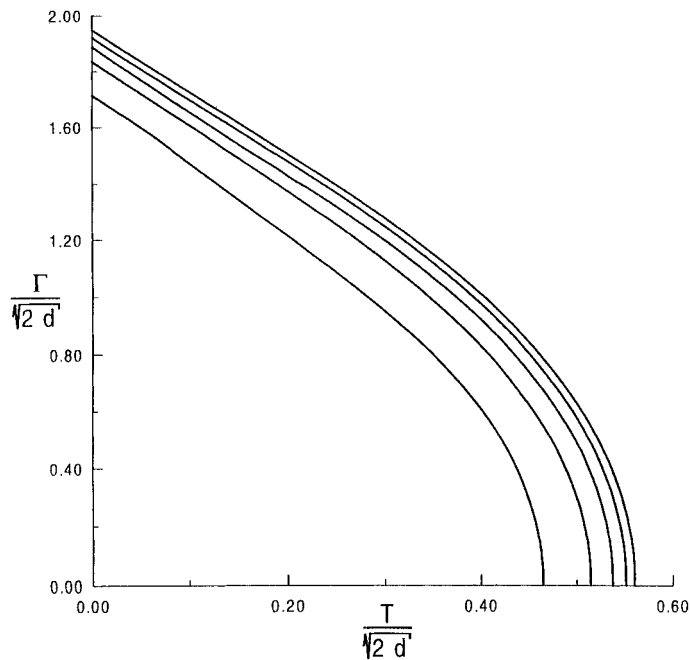

Figure 1. The phase diagram of the classical transverse Ising spin glass with short-range interaction within the Bethe–Peierls approximation for spatial dimensions $d = 2, 3, 4, 5, 6$. The temperature T and transverse field Γ are rescaled by the factor $(2d)^{-1/2}$. The larger the dimension, the higher the corresponding line. Here $J \equiv 1$.

In conclusion, we have studied some relevant aspects of the classical transverse field short-ranged Ising SG in the paramagnetic phase for arbitrary dimensionality d. We expect that our results may also be useful for explaining some properties of the quantum counterpart of the model considered here. However, some questions remain to be answered. For example, on the basis of the general self-consistent equation (7) it is interesting to find solutions with $\mu_{\alpha\alpha'} \neq 0$ in order to see whether the BPA is able to describe correctly our model in the SG phase at arbitrary dimensionalities. Within the present calculations, working in the paramagnetic phase, this is practically impossible, since the complicated integral (12) has been reduced at $T = 0$ to an asymptotic form which is Gaussian-like. Such an asymptoptic form is insufficient when parameters $\mu_{\alpha\alpha'}$ are included even in the replica symmetric form. Therefore, further work will be necessary to elucidate these problems.
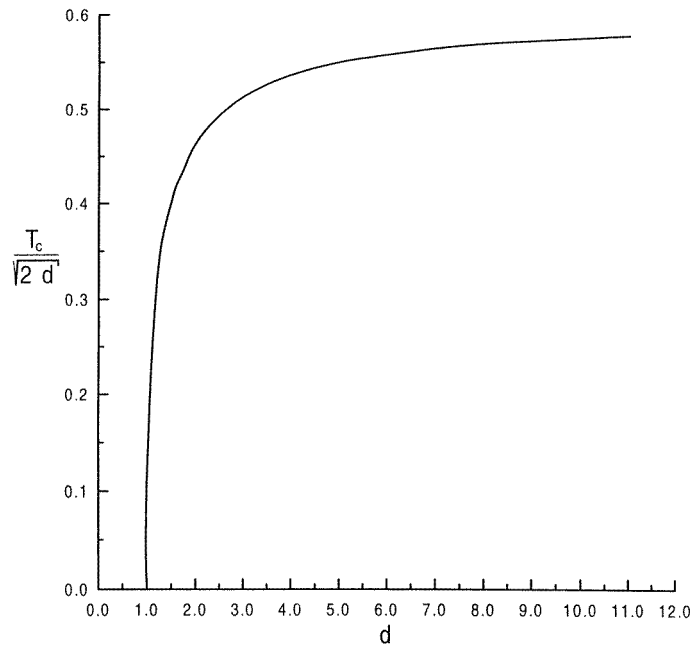
Figure 2. The rescaled critical temperature $T_c(2d)^{-1/2}$ for $\Gamma = 0$ against the dimension d. Here $J \equiv 1$.

Acknowledgments

References

[1] Mezard M, Parisi G and Virasoro M 1988 *Spin Glass Theory and Beyond* (Singapore: World Scientific) and references therein
[2] Fisher K H and Hertz J 1991 *Spin Glasses* (Cambridge: Cambridge University Press)
[3] Binder K and Reger J D 1992 *Adv. Phys.* **41** 547 and references therein
[4] Chakrabarti B K, Dutta A and Sen P 1996 *Quantum Ising Phases and Transitions in Transverse Ising Models* (Berlin: Springer)
[5] Newman C M and Stein D L 1996 *Phys. Rev. Lett.* **76** 515
[6] Parisi G 1996 *Preprint* cond-mat/9603101
[7] Parisi G 1997 *Preprint* cond-mat/9701034
[8] Khurana A 1996 *Phys. Rev.* B **54** 7295
[9] De Dominicis C and Kondor I 1983 *Phys. Rev.* B **27** 606
    Chang M-C 1983 *Solid State Commun.* **47** 529
    Le Doussal P and Harris A B 1989 *Phys. Rev.* B **40** 9249
[10] De Dominicis C, Kondor I and Temesvary T 1991 *J. Phys. A: Math. Gen.* **24** L301
[11] Read N, Sachdev S and Ye J 1995 *Phys. Rev.* B **52** 384
    Read N and Sachdev S 1996 *Nucl. Phys.* B (Proc. Suppl.) **45A** 38

[12] Fisher D S and Huse D A 1986 *Phys. Rev. Lett.* **56** 1601
     Fisher D S and Huse D A 1987 *J. Phys. A: Math. Gen.* **20** L1005
     Bray A J 1988 *Comments Condens. Matter Phys.* **14** 21
[13] Serva M and Paladin G 1996 *Phys. Rev.* E **54** 4637
[14] Bethe H 1935 *Proc. R. Soc.* **150** 552
     Peierls R 1936 *Proc. R. Soc.* **154** 207
[15] Pirc R, Tadić B and Blinc R 1985 *Z. Phys.* B **61** 69
[16] Pirc P, Prevlovšek P and Tadić B 1987 *Solid State Commun.* **64** 137
[17] Li M S, Walasek K and Cieplak M 1997 *Phys. Rev.* B **56** 1
[18] Wu W *et al* 1991 *Phys. Rev. Lett.* **67** 2076
[19] Miller J and Huse D A 1993 *Phys. Rev. Lett.* **70** 3147
[20] Shankar R and Murthy G 1987 *Phys. Rev.* B **36** 336
     McCoy B and Wu T T 1968 *Phys. Rev.* **176** 631
[21] Fisher D S 1992 *Phys. Rev. Lett.* **69** 534